Geometry & Hamiltonian Differential Eq.

A. Kiselev

Leander van Beek

S2715600

EXAM

Onderwerp: Problem 1

Datum:

(1) Let x, y, z be global (e.g. cartesian) coordinates on $\mathbb{R}^3$ and let $a(x,y,z) \in C^2(\mathbb{R}^3 \to \mathbb{R})$. By definition, for any $f, g \in C^{2\ldots\infty}(\mathbb{R}^3 \to \mathbb{R})$, we put

$$\{f,g\} = \left| \frac{D(f,g,a)}{D(x,y,z)} \right| = \begin{vmatrix} f_x & g_x & a_x \\ f_y & g_y & a_y \\ f_z & g_z & a_z \end{vmatrix}$$

Prove that $\{\cdot,\cdot\}$ is a Poisson Bracket.

First, we write down Jacobi's identity for the Poisson bracket:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$$

Working out one term, we use this for symmetry arguments:

$$\{f,g\} = \begin{vmatrix} f_x & g_x & a_x \\ f_y & g_y & a_y \\ f_z & g_z & a_z \end{vmatrix} = f_x(g_y a_z - a_y g_z) + g_x(a_y f_z - f_y a_z) + a_x(f_y g_z - g_y f_z).$$

All subscripts denote derivatives wrt. that variable, such as

$$f_x = \frac{\partial f}{\partial x}$$

From which we see that the Poisson bracket $\{f,g\}_{PB}$ defines a multilinear function of first order partial de.s.

For convenience, let's define $\{f,g\} = \lambda$. Filling in λ into our second poisson bracket

$$\{h,\{f,g\}\} = \{h,\lambda\} = \begin{vmatrix} h_x & \lambda_x & a_x \\ h_y & \lambda_y & a_y \\ h_z & \lambda_z & a_z \end{vmatrix} = h_x(\lambda_y a_z - a_y \lambda_z) - \lambda_x(a_y h_z - h_y a_z)$$
$$a_x(h_y \lambda_z - h_z \lambda_y)$$

Now it starts to get messy:

$$\{h,\lambda\} = \Big\{ h_x \Big[\big[f_x(g_y a_z - a_y g_z) + g_x(a_y f_z - f_y a_z) + a_x(f_y g_z - g_y f_z)\big]_y a_z$$
$$- a_y \big[f_x(g_y a_z - a_y g_z) + g_x(a_y f_z - f_y a_z) + a_x(f_y g_z - g_y f_z)\big]_z \Big] \Big\}$$

+ 2 more terms like in the curly brackets.

We will first only investigate this first term:

$$h_x\Big[\left[f_x(g_y a_z - a_y g_z) + g_x(a_y f_z - f_y a_z) + a_x(f_y g_z - g_y f_z)\right]_y a_z$$

$$- a_y\left[f_x(g_y a_z - a_y g_z) + g_x(a_y f_z - f_y a_z) + a_x(f_y g_z - g_y f_z)\right]_z\Big]$$

In the first half, $[f_x(g_y a_z - a_y g_z) + \dots]$ consists, when fully expanded, out of 6 terms: $f_x g_y a_z - f_x a_y g_z + g_x a_y f_z + \dots$ The partial derivatives ∂_y makes 3 terms from all $f_i g_j a_k$. For this term only, we hence get $6 \cdot 2 \cdot 3 = 36$ terms. Remembering that we only took the first of the three terms of $\{h, \lambda\}$. The same holds for the other two terms, such that we have $2 \cdot 3 \cdot 3 \cdot 6 = 108$ terms for $\{h, \lambda\}$ with second order derivatives. This hole process was independent of $\{h, f, g\}$, so for the other two terms in the Jacobi identity we can do the same. In total, we then have $2 \cdot 3 \cdot 3 \cdot 3 \cdot 6 = 324$ terms with a second order differential.

On to the second part of the problem; show that in the left hand side of the Jacobi identity all terms with a second time derivative of f, g, h or a cancel out.

We will again only need one term to show that this is true. Take for example $\{g, \{h, f\}\} + \{h, \{f, g\}\}$ which contains all second order terms of f.

(This term is equal to $-\{f,\{g,h\}\}$). We know we can also write this term as $\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$, such that we can define the differential operator

$$D_1(\varphi) = \{g,\varphi\} \quad \text{and} \quad D_2(\varphi) = \{h,\varphi\}$$

Now we rewrite

$$\{g,\{h,f\}\} + \{h,\{f,g\}\} = \{g,\{h,f\}\} - \{h,\{g,f\}\}$$

by the anticommutativity of the Poisson Bracket: $\{f,g\} = -\{g,f\}$.

$$\Rightarrow \{g,\{h,f\}\} - \{h,\{g,f\}\} = D_1(D_2(f)) - D_2(D_1(\varphi))$$

Since we are dealing with operators:

$$\Rightarrow (D_1 D_2 - D_2 D_1) f$$

Now by the properties of D_1 and D_2, we can complete the proof.

$$D_1 = \sum_k \xi_k \frac{\partial}{\partial x_k}, \quad D_2 = \sum_k \eta_k \frac{\partial}{\partial x_k}$$

For the products D_1D_2 and D_2D_1 we then obtain (by applying the chain rule)

$$D_1 D_2 = \sum_{k,l} \xi_k \eta_l \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{k,l} \xi_k \frac{\partial \eta_l}{\partial x_k}\frac{\partial}{\partial x_l}$$

$$D_2 D_1 = \sum_{k,l} \eta_k \xi_l \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{k,l} \eta_k \frac{\partial \xi_l}{\partial x_k}\frac{\partial}{\partial x_l}$$

Such that the difference between them is

$$D_1 D_2 - D_2 D_1 = \sum_{k,l} \left(\xi_k \frac{\partial \eta_l}{\partial x_k} - \eta_k \frac{\partial \xi_l}{\partial x_k} \right) \frac{\partial}{\partial x_l}$$

We see that there is no possibility for double derivatives of f to show up; hence they have all cancelled. The process can be repeated to show the same for g and h.

To prove that the given expression is actually a Poisson bracket, we still need to show that the bracket is skew symmetric, $\{f,g\} = -\{g,f\}$.
This is easy, as it translates in the permutation of the left and middle column of the determinant, which imposes a minus sign

$$\begin{vmatrix} f_x & g_x & a_x \\ f_y & g_y & a_y \\ f_z & g_z & a_z \end{vmatrix} = - \begin{vmatrix} g_x & f_x & a_x \\ g_y & f_y & a_y \\ g_z & f_z & a_z \end{vmatrix}$$

Finally, we need to show that Leibniz' rule holds: $\{fg, h\} = f\{g,h\} + g\{h,f\}$.
This is also easily shown through

$$\begin{vmatrix} (fg)_x & h_x & a_x \\ (fg)_y & h_y & a_y \\ (fg)_z & h_z & a_z \end{vmatrix} = \begin{vmatrix} gf_x + fg_x & h_x & a_x \\ gf_y + fg_y & h_y & a_y \\ gf_z + fg_z & h_z & a_z \end{vmatrix}$$

$$= g \begin{vmatrix} f_x & h_x & a_x \\ f_y & h_y & a_y \\ f_z & h_z & a_z \end{vmatrix} + f \begin{vmatrix} g_x & h_x & a_x \\ g_y & h_y & a_y \\ g_z & h_z & a_z \end{vmatrix}$$

$$= g\{f,h\} + f\{g,h\}$$

And hence the bracket is a poisson bracket.
The fact that $\{\cdot,\cdot\}$ is bilinear is just a special case of Leibniz rule, where one of the f, g is not a function, but a constant.

Onderwerp: Problem 2

Datum:

Let x, y, z be cartesian coordinates in $\mathbb{R}^3$ and $\vec{A} = -Hy \cdot \vec{e}_x$ be the "vector potential" of magnetic field $\vec{\mathcal{H}} = \vec{\nabla} \times \vec{A} = \text{rot } \vec{A}$.

Show that this is a constant and homogeneous magnetic field.

a/ We can easily compute $\vec{\mathcal{H}}$:

$$\vec{\mathcal{H}} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z)$$
$$+ \hat{z}(\partial_x A_y - \partial_y A_x)$$

Where the only remaining term is $-\hat{z}\partial_y A_x = H\hat{z}$, here $\hat{z}$ denoting a unit vector. We can already see that the magnetic field is homogeneous and by applying

$$\vec{\nabla} \cdot \vec{\mathcal{H}} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

we know this for sure, as the divergence signifies the tendency of the field to "compactify" or "stretch".

b/ Find the complete integral of the "Hamilton-Jacobi Equation:

$$\mathcal{E}_{HJ} = \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x} - \frac{e_0}{c}\mathcal{H}y\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] = 0 \right\}$$

We notice that there is no direct time dependence in the equation, such that we may write $S = W - Et$. Because the action does not 'feel' addition of a constant, we may say

$$\left(\frac{\partial S}{\partial x} - \frac{e_0}{c}\mathcal{H}y\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = 2mE$$

In this equation, we also see no direct dependence on the x- and z-coordinate, so we know that these are cyclic. This means that we replace $\frac{\partial S}{\partial x} = \alpha_x$:

$$\left(\alpha_x - \frac{e_0}{c}\mathcal{H}y\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \alpha_z^2 = 2mE$$

Which we rewrite to the form

$$\left(\frac{\partial S}{\partial y}\right)^2 = 2mE - \alpha_z^2 - \left(\alpha_x - \frac{e_0}{c}\mathcal{H}y\right)^2$$

$$\int \partial S = \int \sqrt{[2mE - \alpha_z^2] - \left(\alpha_x - \frac{e_0}{c}\mathcal{H}(y)\right)^2}\, dy$$

This looks like a nasty expression but is actually a standard integral of the form $\int \sqrt{1-x^2}$ after we apply some substitutions.

$$\frac{1}{(2mE - \alpha_z^2)^{1/2}} \int \sqrt{1 - \frac{\left(\alpha_x - \frac{e_0}{c}\mathcal{H}y\right)^2}{2mE - \alpha_z^2}}\, dy$$

$$\frac{1}{(2mE - \alpha_z^2)^{1/2}} \int \sqrt{1 - u^2}\, \frac{dy}{du}\, du$$

Where we have put $u = \frac{\alpha_x - \frac{e_0}{c}Hy}{\sqrt{2mE - \alpha_z^2}} \rightarrow \frac{du}{dy} = \frac{-\frac{e_0}{c}H}{\sqrt{2mE - \alpha_z^2}}$

We obtain

$$\frac{\sqrt{2mE - \alpha_z^2}}{\sqrt{2mE - \alpha_z^2}} \frac{1}{\frac{e_0}{c}H} \int \sqrt{1 - \sin^2(t)}\, \frac{du}{dt}\, dt =$$

$$= \frac{1}{\frac{e_0}{c}H} \int \cos^2(t)\, dt = \int \left[\frac{1}{2} + \frac{\cos(2t)}{2}\right] dt$$

Onderwerp: Problem 2

Datum:

Which becomes

$$\frac{1}{\frac{e_0}{c}H}\left[\frac{t}{2}+\frac{2\sin(2t)}{4}\right]=$$

By the last substitution, we have $u=\sin(t)$ and therefore $t=\arcsin(u)$. We see:

$$=\frac{1}{\frac{e_0}{c}H}\left[\frac{\arcsin(u)}{2}+\frac{\sin(\arcsin(u)\cdot 2)}{4}\right]$$

$$\sin(\arcsin(u)\cdot 2)=2\sin(\arcsin(u))\cos(\arcsin(u))$$
$$=2u\cos(\arcsin(u))$$
$$=2u\sqrt{1-\sin^2(\arcsin(u))}=2u\sqrt{1-u^2}$$

Back to our integral:

$$\frac{1}{\frac{e_0}{c}H}\left[\frac{\arcsin(u)}{2}+\frac{2u\sqrt{1-u^2}}{4}\right]$$

So finally:

$$S_y=\frac{1}{\frac{e_0}{c}H}\left[\frac{\arcsin\left(\frac{\alpha_x-\frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z^2}}\right)}{2}+\frac{2}{4}\,\frac{\alpha_x-\frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z^2}}\sqrt{1-\frac{\left(\alpha_x-\frac{e_0}{c}Hy\right)^2}{2mE-\alpha_z^2}}\right]$$

Which is our complete integral of E_{HJ}.

b) From the Hamilton-Jacobi equation we know

$$S=W+Et$$

$$\cancel{H=\Big(\underbrace{p_x}_{\alpha_x}-\frac{e_0}{c}Hy\Big)^2+\Big(\frac{\partial S}{\partial y}\Big)^2+\underbrace{p_z^2}_{\alpha_z}=E}\qquad \Big(\frac{\partial W}{\partial y}\Big)^2$$

$$\cancel{\frac{\partial W}{\partial y}=\sqrt{\Big(p_x-\frac{e_0}{c}Hy\Big)^2+(\dots)-p_z-E}}$$

With our current

$$W = \frac{1}{\frac{e_0}{c}\mathcal{H}}\left[\frac{\arcsin\left(\frac{\alpha_x - \frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z^2}}\right)}{2} + \frac{1}{2}\frac{\alpha_x - \frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z^2}} \times \sqrt{1-\frac{(\alpha_x - \frac{e_0}{c}Hy)^2}{2mE-\alpha_z^2}}\right]$$

We can find the constants β through

$$\beta_x = \frac{\partial W}{\partial \alpha_x} = \left(\frac{e_0}{c}\mathcal{H}\right)^{-1}\left[\frac{1}{\sqrt{1-\left(\frac{\alpha_x - \frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z^2}}\right)^2}}\frac{1}{2\sqrt{2mE-\alpha_z^2}}\right.$$

From chain rule to α_x

$$+\frac{1}{2}\frac{1}{\sqrt{2mE-\alpha_z^2}}\sqrt{1-\frac{(\alpha_x - \frac{e_0}{c}Hy)^2}{2mE-\alpha_z^2}}$$

$$\left.+\frac{1}{2}\frac{\alpha_x - \frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z^2}}\left(\sqrt{1-\frac{(\alpha_x - \frac{e_0}{c}Hy)^2}{2mE-\alpha_z^2}}\right)^{-1} 2\frac{(\alpha_x - \frac{e_0}{c}Hy)}{(2mE-\alpha_z^2)^2}\right]$$

$$\beta_z = \frac{\partial W}{\partial \alpha_z} = \left(\frac{e_0}{c}\right)^{-1}\frac{1}{\mathcal{H}}\left[\frac{1}{2}\frac{1}{\sqrt{1-\frac{\alpha_x - \frac{e_0}{c}Hy}{(2mE-\alpha_z^2)}}}\frac{+2\alpha_z(\alpha_x - \frac{e_0}{c}Hy)}{(2mE-\alpha_z^2)^2}\right.$$

$$-\frac{1}{4}\frac{(\alpha_x - \frac{e_0}{c}Hy)}{(\sqrt{2mE-\alpha_z^2})^3}(-2\alpha_z)\sqrt{1-\frac{(\alpha_x - \frac{e_0}{c}Hy)^2}{\sqrt{2mE-\alpha_z^2}}}$$

$$\left.+\frac{1}{4}\left(\sqrt{1-\frac{(\alpha_x - \frac{e_0}{c}Hy)^2}{2mE-\alpha_z^2}}\right)^{-1}\frac{-2\alpha_z(\alpha_x - \frac{e_0}{c}Hy)}{(2mE-\alpha_z^2)^2}\right]$$

$$t = \frac{\partial W}{\partial E} = \frac{1}{\frac{e_0}{c}\mathcal{H}}\left[\frac{1}{2}\frac{1}{\sqrt{1-\frac{\alpha_x - \frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z}}}}\frac{\frac{e_0}{c}yH}{\sqrt{2mE-\alpha_z}}\left(\frac{1}{2}\right)2m\right.$$

$$+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{2m(\alpha_x - \frac{e_0}{c}Hy)}{\sqrt{2mE-\alpha_z^2}}\sqrt{1-\frac{(\alpha_x - \frac{e_0}{c}Hy)^2}{2mE-\alpha_z^2}}$$

How to get to an expression for $q_i p_i$ for the Helix I could not see.

$$\left.+\frac{1}{2}\frac{\alpha_x - \frac{e_0}{c}Hy}{\sqrt{2mE-\alpha_z^2}}\cdot\frac{1}{2}\left(\sqrt{1-\frac{(\alpha_x - \frac{e_0}{c}Hy)^2}{2mE-\alpha_z^2}}\right)^{-1}\frac{2m(\alpha_x - \frac{e_0}{c}Hy)^2}{(2mE-\alpha_z^2)^2}\right]$$

Onderwerp: Problem 3

Datum:

In Kepler's problem of orbital motion, consider the shifted potential $U = -\frac{\alpha}{r} + \delta U(r)$, where

a/ $\delta U(r) = \frac{\beta}{r^2}$, $\beta \to 0$

b/ $\delta U(r) = \frac{\gamma}{r^3}$, $\gamma \to 0$

The trajectories are no longer closed ellipsis; in one period $\left(\text{in } r\Big|_{r_{min}}^{r_{max}}\right)$, the perihelion is shifted by $\Delta\varphi$. Find $\Delta\varphi$, (assuming $\beta \to 0$, $\gamma \to 0$).

We write down the Lagrangian to find constants of motion:

$$\mathcal{L} = \underbrace{\frac{1}{2} m\left(\dot{r}^2 + (r\dot{\theta})^2\right)}_{T} + \underbrace{\frac{\alpha}{r} - \frac{\beta}{r^2}}_{U(r)+\delta U}$$

We work out ε_{EL}

$$\theta \left\{ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m r^2 \dot{\theta} = \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow m r^2 \dot{\theta} = M_z, \text{ constant of motion} \right.$$

$$r \left\{ \begin{array}{l} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r} \\ \frac{\partial \mathcal{L}}{\partial r} = m r \dot{\theta}^2 - \frac{\alpha}{r^2} + \frac{2\beta}{r^3} \end{array} \right\} \Rightarrow m\ddot{r} = m r \dot{\theta}^2 - \frac{\alpha}{r^2} + \frac{2\beta}{r^3}$$

Which we can rewrite into

$$m\ddot{r} = \frac{M_z}{m r^3} - \frac{\alpha}{r} + \frac{2\beta}{r^3}$$

When we write out the total energy of the system, we obtain a useful expression.

$$E = \frac{m\dot{r}^2}{2} + \frac{M_z^2}{2mr^2} - \frac{\alpha}{r} + \frac{\beta}{r^2}$$

As $\dot{r} = dr/dt$, we see

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - \frac{M_z^2}{2mr^2} + \frac{\alpha}{r} - \frac{\beta}{r^2}\right]}$$

We notice that this differential equation is separable:

$$\int dt = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{M_z^2}{2mr^2} + \frac{\alpha}{r} - \frac{\beta}{r^2}\right)}} = t$$

Now comes a trick we borrowed from Landau & Lifshitz, the integrand can be written as a differential in M_z. We substitute $dt = \frac{mr^2}{M_z} d\varphi$ which we can do by our $\mathcal{E}_{EL}$ in θ, as we found that $m r^2 \frac{d\varphi}{dt} = M_z$.

$$\Rightarrow \int dt = \int \frac{mr^2}{M_z} d\varphi = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{M_z^2}{2mr^2} + \frac{\alpha}{r} - \frac{\beta}{r^2}\right)}}$$

$$\Rightarrow \varphi = \int \frac{M_z/r^2}{\sqrt{2m\left(E - \frac{M_z^2}{2mr^2} + \frac{\alpha}{r} - \frac{\beta}{r^2}\right)}} dr \quad (*)$$

Now our integrand is

$$-2\frac{\partial}{\partial M_z}\left\{\sqrt{2m\left(E - \frac{M_z^2}{2mr^2} + \frac{\alpha}{r} - \frac{\beta}{r^2}\right)}\right\}$$

Which if we substitute

$$u = 2m\left(E - M_z/mr^2 + \alpha/r - \beta/r^2\right)$$

and apply the chain rule, gives us exactly our own expression

$$-2\left(\frac{\partial M_z}{\partial u}\right)^{-1}\frac{\partial}{\partial u}\left(\sqrt{u}\right) = -2\,\frac{1}{2}\left(-\frac{2M_z}{mr^2}\,\frac{1}{\sqrt{u}}\right) \quad (*)$$

Our integral becomes, after interchanging integration and differentiation

$$\varphi = \frac{\partial}{\partial M_z}\int -2\sqrt{2m\left(E - \frac{M_z^2}{2mr^2} + \frac{\alpha}{r} - \frac{\beta}{r^2}\right)}\, dr$$

Onderwerp: Problem 3

Datum:

$$\varphi = -2\frac{\partial}{\partial M}\int \sqrt{2m\left(\left(E - U(r)\right) - \frac{M^2}{r^2}\right)}\, dr$$

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2} = U + \delta U$$

We Taylor expand this expression in δU up to linear terms: around $\delta U = 0$:

$$\sqrt{2m\left(E - U + \delta U\right) - \frac{M_z^2}{r^2}} = \frac{2m\,\delta U}{\sqrt{2m(E-U) + -\,M_z^2/r^2}} \quad \leftarrow \text{linear term (order 1)}$$

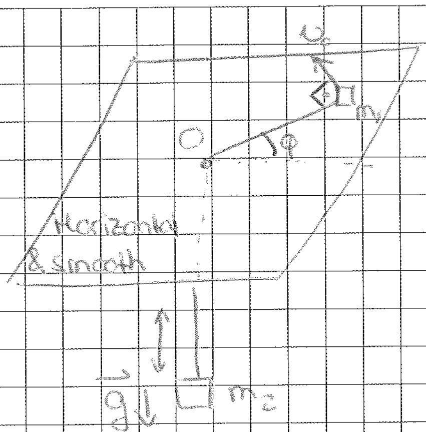
$$+ \sqrt{2m(E-U) - \frac{M_z}{r^2}} \quad \leftarrow \text{Constant term (order 0)}$$

$$+ \bar{O}(2)$$

The constant term gives us 2π, as for the unperturbed orbit we have $E = U$. How to integrate the linear term in δU I could not find out. We need a parameterization mapping r to φ such that we can change the integration variable

Onderwerp: Problem 4

Datum:



Prove that the domain of motion of m_1 is the annulus bounded by two concentric circles of radii r_0 and $r_1 = 8$, $r_1 > 0$

We start off by writing down the Lagrangian.

$$\mathcal{L} = \frac{1}{2}\left(m_1 \dot r^2 + m_1 r^2 \dot\theta^2 + m_2 \dot r^2\right) + mgr$$

Using $\mathcal{E}_{LL}$, we see

$$\theta \left\{ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = m_1 r^2 \dot\theta = 0 \Rightarrow M_z = m_1 r^2 \dot\theta \text{ is a cons. of motion} \right.$$

We do the same for r

$$r \left\{ \begin{array}{l} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) = \frac{d}{dt}\left(m_1 \dot r + m_2 \dot r\right) = (m_1 + m_2)\ddot r \\ \frac{\partial \mathcal{L}}{\partial r} = m_1 r \dot\theta^2 + m_2 g \end{array} \right.$$

Combining the two: $(m_1 + m_2)\ddot r = m_1 r \dot\theta^2 + m_2 g$

We can also write the total energy of the system

$$H = \frac{1}{2}(m_1 + m_2)\dot r^2 + \frac{1}{2} m_1 r^2 \dot\theta^2 - mgr$$

$$E = \frac{p_r^2}{2\mu} + \frac{p_\varphi^2}{2 m_1 r^2} + m_2 g r$$

Where we have set $\mu = m_1 + m_2$. Upon multiplying with r^2, we find a 3^{rd} order polynomial.

$$E r^2 = \frac{p_r r^2}{2\mu} + \frac{p_\varphi}{2 m_1} + m_2 g r^3$$

$$\Rightarrow m_2 g r^3 + \left(\frac{p_r}{2m} - E\right) r^2 + \frac{p_\varphi}{2 m_1} = 0.$$

Because we are at an extremum for the r-coordinate,

Onderwerp: Problem 4

Datum:

$\dot{r}$ and by extension $p_r = 0$

$$\Leftrightarrow \quad m_2 g r^3 + -E r^2 + \frac{p_\varphi}{2m_1} = 0$$

Which is a polynomial we can solve by Viet's theorem.

Viet's theorem states that for a polynomial of order n, the polynomial can be rewritten into

$$(a_1 - x)(a_2 - x) \dots (a_n - x) = 0$$

Where a_i are the roots of the polynomial.

For $n=3$, this gives

$$(a-x)(b-x)(c-x) = 0$$

$$\Leftrightarrow (ab - ax - bx + x^2)(c - x)$$

$$= abc - abx - acx + ax^2 - bcx + bx^2 + cx^2 - x^3$$

$$= -x^3 + (a+b+c)x^2 + (-ab - ac - bc)x + abc$$

We can write our polynomial in a similar form

$$\Leftrightarrow -r^3 + \frac{E}{m_2 g} r^2 - \frac{p_\varphi}{2m_1 m_2 g} = 0$$

Polynomials are equal when their coefficients are equal. Hence we can state:

① $a + b + c = \frac{E}{m_2 g}$

② $ab + ac + bc = 0$

③ $abc = -\frac{p_\varphi}{2m_1 m_2 g}$

We know one solution of this polynomial to be r_0. We set $a = r_0$ and plug this into ②

Onderwerp: Problem 4

Datum:

$$ab + ac + bc = 0 \Leftrightarrow r_0 b + r_0 c + bc = 0$$

Now we can express c in b and v.v.:

$$r_0 b + r_0 c + bc = r_0 b + c(r_0 + b) = 0$$

$$\Leftrightarrow c = \frac{-r_0 b}{r_0 + b}$$

Replacing c by this expression in (1) gives

$$r_0 + b + \frac{-r_0 b}{r_0 + b} = \frac{E}{m_2 g}$$

$$\Leftrightarrow r_0^2 + b r_0 + b r_0 + b^2 - r_0 b = \frac{E r_0}{m_2 g} + \frac{E b}{m_2 g}$$

Taking all terms in b to one side

$$\underbrace{1}_{A} b^2 + b \underbrace{\left(r_0 - \frac{E\cancel{r_0}}{m_2 g}\right)}_{B} + \underbrace{\left(r_0^2 - \frac{E r_0}{m_2 g}\right)}_{C} = 0$$

Which is an order-2 polynomial, solvable by the quadratic formula

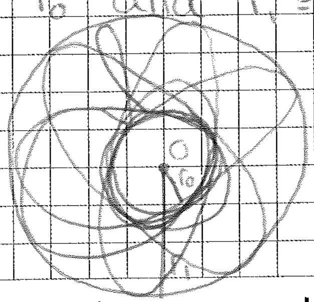
$$D = B^2 - 4AC = \left(r_0 - \frac{E}{m_2 g}\right)^2 - 4\left(r_0^2 - \frac{E r_0}{m_2 g}\right)$$

Now the solutions of the quadratic formula actually give the sought radius r_1 and an additional solution,

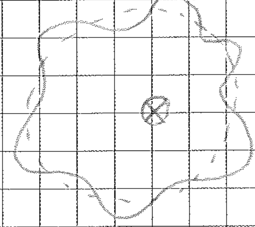
$$\{r_1, r_2\} = \frac{-B \pm \sqrt{D}}{2A} = \frac{-\left(r_0 - \frac{E}{m_2 g}\right) \pm \sqrt{\left(r_0 - \frac{E}{m_2 g}\right)^2 - 4\left(r_0^2 - \frac{E r_0}{m_2 g}\right)}}{2}$$

So the other root of our system was

$$r_0 \text{ and } r_1 = \frac{E}{2 m_2 g} - \frac{r_0}{2} + \frac{1}{2}\sqrt{\left(r_0 - \frac{E}{m_2 g}\right)^2 - \left(4 r_0^2 - \frac{E r_0}{m_2 g}\right)}$$



A satellite was orbiting the earth along a circular orbit, which then changed such that the satellite began to oscillate around the old trajectory. Find $\omega(a_0)$, a_0 being the amplitude of the oscillations



We start again by constructing the Lagrangian, $\mathcal{L}$.

$$\mathcal{L} = \frac{1}{2} m \left[\dot{r}^2 + (r\dot{\theta})^2 \right] - V(r)$$

Because our potential is only a function of r, we can already work out Euler-Lagrange for θ:

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt} m r^2 \dot{\theta} \\ &\frac{\partial \mathcal{L}}{\partial \theta} = 0 \end{aligned}\right\} \Rightarrow m r^2 \dot{\theta} = M_z \text{ constant.}$$

We set $V(r)$ to be the gravitational potetial, $V = -mgr$, where $g = \frac{GM}{R_0^2}$ such that $V = -\frac{mGM}{r}$

To get it into a form similar to the exam, we have to rewrite: $+\frac{GMm}{r} = +\frac{GMm}{r R_0^2} R_0^2 = +\frac{GM}{R_0^2} \frac{m R_0^2}{r} = +\frac{g m R_0^2}{r}$

Using this for E_L on r:

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{d}{dt}\left[m\dot{r} \right] = m\ddot{r} \\ &\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\theta}^2 + + \frac{g m R_0^2}{r^2} \end{aligned}\right\} m\ddot{r} = m r \dot{\theta}^2 + \frac{g m R_0^2}{r^2}$$

Now $m\ddot{r}$ is supposed to be equal to $\frac{\partial U}{\partial r}$, so let's integrate the right hand side.

Claim $mr\dot{\theta}^2 + \frac{gmR_0^2}{r} = \frac{\partial u}{\partial r}$

$$\int du = \int\left[mr\dot{\theta}^2 + \frac{gmR_0^2}{r^2}\right]dr = \frac{1}{2}mr^2\dot{\theta}^2 - \frac{gmR_0^2}{r} + C$$

C being the integration constant. Now actually, as $M_z = mr^2\dot{\theta}$, such that $mr^2\dot{\theta}^2 = M_z^2/(mr^2)$, we see

$$U(r) = \frac{M_z^2}{2mr^2} - \frac{gmR_0^2}{r} + C$$

So our claim is correct, up to an integration constant. $U(r)$ is our effective potential.

From classical mechanics, we know that $M_z = m\omega_0 R^2$.
Substituting this in the derivative $\frac{\partial u}{\partial r}$ we are asked to expand:

$$\left(-\frac{\partial U}{\partial r}\right) = 2m^2\omega_0^2 R^4 \frac{1}{2mr^3} + \frac{mgR_0^2}{r^2}$$

$$= -\frac{m\omega_0^2 R^4}{r^3} + \frac{mgR_0^2}{r^2}$$

By the centripetal force, $mg = mv^2/r = \frac{mr^2\dot{\theta}^2}{r}$
such that $g = r^2\dot{\theta}^2/r = \omega_0^2 r$, where $r \to R_0$ as it is the gravitational acceleration at the surface

$$-\frac{\partial u}{\partial r} = -\frac{m\omega_0^2 R_0^4}{r^3} + \frac{m\omega_0^2 R_0^3}{r^2} = m\omega_0^2 R_0^2\left(-\frac{R_0^2}{r^3} + \frac{R_0}{r^2}\right)$$

We know the Taylor expansion to be (to order m)

$$T_m(x) = \sum_{n=0}^{m} \frac{1}{n!} f^n(x_0)x^n = f(x_0) + f'(x_0)x + \frac{1}{2}f''(x_0) + \ldots$$

~~Setting $r = R + x$ as instructed~~

Writing down some derivatives around R_0 for $\frac{\partial u}{\partial r}$.

$$\frac{\partial}{\partial r}\left[-\frac{\partial U}{\partial r}\right]\Bigg|_{r=R_0} = m\omega_0^2 R_0^2\left(\frac{3R_0^2}{r^4} - \frac{2R}{r^3}\right)\Bigg|_{r=R_0} = m\omega_0^2 R_0^2\left(\frac{3}{R_0^2} - \frac{2}{R_0^2}\right) = m\omega_0^2$$

$$\frac{\partial^2}{\partial r^2}\left[-\frac{\partial U}{\partial r}\right]\Bigg|_{r=R_0} = m\omega_0^2 R_0^2\left(\frac{-12R_0^2}{r^5} + \frac{6R_0}{r^4}\right)\Bigg|_{r=R_0} = m\omega_0^2 R_0^2\left(-\frac{12}{R_0^3} + \frac{6}{R_0^3}\right) = -\frac{m\omega_0^2 6}{R_0}$$

$$\frac{\partial^3}{\partial r^3}\left[-\frac{\partial U}{\partial r}\right]\Bigg|_{r=R_0} = m\omega_0^2 R_0^2\left(\frac{+60R_0^2}{r^6} + \frac{-24R_0}{r^5}\right)\Bigg|_{r=R_0} = m\omega_0^2 R_0^2\left(\frac{60}{R_0^4} - \frac{24}{R_0^4}\right) = \frac{36m\omega_0^2}{R_0^2}$$

Filling these in in the Taylor expansion:

$$T(x) = -m\omega_0^2 x + \frac{6}{2!}\frac{m\omega_0^2}{R_0}x^2 - \frac{36}{3!}\frac{m\omega_0^2}{R_0^2} + \bar{o}(x^3)$$

Where $\frac{\partial U}{\partial r}\Big|_{r=R_0} = 0$, so the Taylor series starts with the linear term. We have also set $x = r - R \Rightarrow r = R + x$. This poses no problems in derivatives as $\dot{x} = -\cancel{\dot{R}} + \dot{r} = \dot{r}$.

Putting all this in $m\ddot{r} = -\frac{\partial U}{\partial r}$, we find

$$m\ddot{x} = -m\omega_0^2 x + \frac{6}{2}\frac{m\omega_0^2}{R_0^2}x^2 - 6\frac{m\omega_0^2}{R_0^2}x^3 + \bar{o}(x^3)$$

$$\Rightarrow \quad \ddot{x} + \omega_0^2 x = 3\frac{m\omega_0^2}{R_0^2}x^2 - 6\frac{m\omega_0^2}{R_0^2}x^3 + \bar{o}(x^3).$$